On The Source of Mathematical Intuition

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Introduction

The main question we here address is the following: how can we, human beings, deduce without any apparent recourse to experience, i.e. seemingly *a priori*, truths about the universe in which we live in? And what is the source of the intuitions that guide us in the quest of those truths? Our main focus here are *mathematical* truths. Therefore, we also have to deal with what exactly is a *mathematical truth*.

Let us look at an example that will be used throughout the paper. Consider a *graph*, which is just a

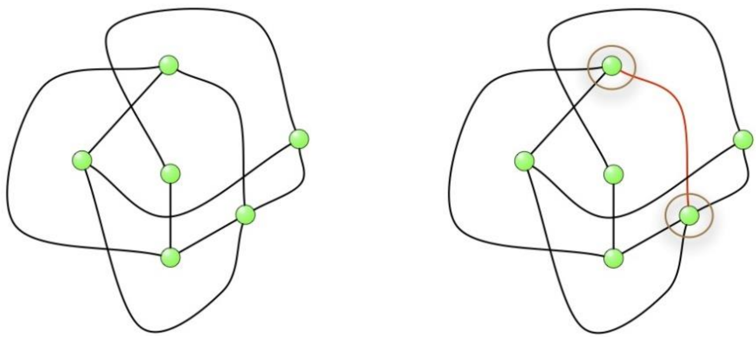

Fig. 1: A graph

set of points, some of which are connected by lines (see the left side of Fig.

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1). The shape of the lines is irrelevant, all that matters is whether two points are or not connected by a line. In this context one usually calls *vertices* to the points, while the lines are called *edges*. The number of lines (edges) that come out a given point (vertex) is called the *degree* of that point (vertex). It is very well-known, and easy to see, that the following proposition, which we will henceforth call *P*, holds true:

Proposition *P*: *In any graph, the sum of the degrees of all vertices is equal to the double of the number of edges.*

The reason is simply that when one adds all degrees, one is adding all lines twice, since a line comes out of exactly two vertices (see the right side of Fig. 1).

Now, this implies, for instance, that one can never connect five things so that each one is connected to precisely other three. Be it five branch offices of a business that someone wants to connect with fiber optic cables so that each one is connected to exactly other three, or five capitals that an airline company wants to connect with flights so that from each capital one can fly to exactly other three, or simply five rocks that one wants to connect with ropes so that each stone is connected to exactly other three, it just cannot be done! Why? Because in each one of these tasks one is looking for

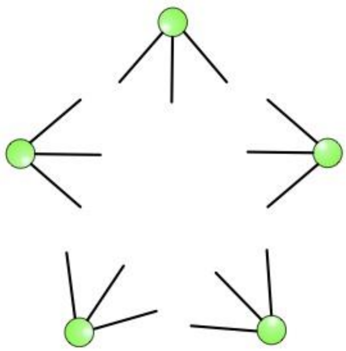

Fig. 2: It cannot be done!

something that is equivalent to the construction of a graph in which the sum of the degrees of all vertices is 15, an odd number, which cannot be, by proposition *P*.

In this way, one has concluded, without any doubt whatsoever, that a very great number of tasks cannot be done, and that was accomplished without no need to make a single experiment. How is this possible? And where do the ideas and intuitions behind the argument come from?

1. What exactly is Mathematics?

It is not easy to define what Mathematics is about. To simply say that it is the "science of numbers" is so vague and inaccurate as saying that literature is the "art of letters". In the first place, Mathematics deals not only with numbers, but with a rather extensive panoply of objects like geometric figures, sets, functions, algebraic structures, topological spaces, graphs, and so on, some of which have no connection to numbers. Secondly, as with literature, where one merely uses letters to convey thoughts and feelings, in Mathematics, when numbers are used, is mostly to convey thoughts and relationships. Note that, although numbers intervene in the statement of Proposition *P*, this proposition is not about numbers, but about some relationship among the number of points and the number of lines.

Sometimes, it is said that Mathematics is a language, which one then claims to be universal, often comparing it with music. But the really interesting question is what does this language expresses What does Mathematics study? It is more or less clear that, roughly, Physics studies the laws of interaction of matter and energy; that Chemistry studies the interaction of molecules and the properties of the compounds that they form; that Biology studies living organisms, mainly their internal organization, and that Ethology studies the external, individual and social, behavior of complex living organisms. But, even roughly, what part of reality does Mathematics study? Or, is it the case that it does not study anything real? But then, how to explain its truly amazing descriptive and, especially, its predictive power?

A striking example of both these powers is the discovery by James Clerk Maxwell (1831--1879), on paper (!) and with the paramount help of mathematics, around 1864, of electromagnetic waves. He realized that light is such a wave, and that there are many more kinds of these waves, whose existence was only experimentally confirmed more than two decades later, by Heinrich Hertz (1857--1894),[1] who wrote:

> It is impossible to study this wonderful theory without feeling as if the mathematical equations had an independent life and an intelligence of their own, as if they were wiser than ourselves, indeed wiser than their discoverer, as

[1] See [3], Chap. XX, and [4], Chap. 6.

> if they gave forth more than he had put into them... [2]

Other examples would be: the discovery, in 1900, of the quantum nature of the atomic world, by Max Planck (1858--1947), who was literally forced by mathematics to accept physical interpretations he did not like in the least;[3] Riemannian geometry, developed by Bernhard Riemann (1826--1866), around 1854, inspired by the work of Gauss (1777--1855), and later elaborated by Beltrami (1835--1900), Christoffel (1829--1900), Lipschitz (1832--1903), Ricci (1853--1925) and Levi-Civita (1873--1941), which played, more than half a century later, a crucial role in the general theory of relativity of Albert Einstein (1879--1955);[4] the prediction in 1928 of anti-matter made by Paul Dirac (1902--1984),[5] which was experimentally confirmed four years later.

These are just some examples of what has been called by the physicist Eugene Wigner[6] the "unreasonable effectiveness of Mathematics in the Natural Sciences".[7] This unreasonable effectiveness does show that whatever the language of Mathematics expresses, it must have some real content. This has been eloquently articulated by Galileo, in a famous passage of his 1623 book *Il Saggiatore* (Chap. 6):

> Philosophy is written in this grand book - I mean the universe - which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it.

It is also often said of Mathematics that its deals only with approximations to reality. That since perfect triangles or perfect circles do not exist, results about triangles or circles can only be used within prescribed degrees of error. I will return to this later, but now I just wish to note that Proposition *P* presented above is not an approximation to reality, but an exact description of a feature of reality. It is indeed utterly impossible to connect five rocks, or any other five things, so that anyone of them is connected to exactly three other, or in any other way which violates what that proposition states. The relation

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[2] Quoted in [5], p. 101.
[3] 6], p. 4.
[4] See [7], Chap. 37, and §4 of Chap. 48.
[5] See [8], p. 392.
[6] Nobel laureate in Physics, 1963.
[7] In [9]. See also [11,12].

stated by the proposition is absolutely necessary, as all of its instances. But the interesting point is that, while one can easily imagine, and even conceptually play with worlds that have different physical laws, like one in which the gravitation law would be inversely proportional to the cube of the distance, instead of the square, one cannot envision an universe where Proposition *P* is false. As Raymond Smullyan writes in [13, p. 47]:

> The physical sciences are interested in the state of affairs that holds for the actual world, whereas pure mathematics and logic study all possible state of affairs.

This, to me, hints at the fact that, in Mathematics, one is studying some sort of deep structural laws on which the universe is built upon, something that underlies the physical laws of Nature, maybe even some structural laws that must be satisfied by any possible Universe. Pushing the point a bit further, it does seem that when the Universe come into existence at the so called "Big Bang" (a not very good name, by the way), it already come with some structural fabric and that, somehow, Mathematics is the area of human knowledge that studies precisely that fabric, a sort of logic inner fabric underlying everything.

2. Mathematical Objects

The ontological status of mathematical objects has been a source of philosophical debate since, at least the time of Plato. Whether mathematical objects are real or ideal, are discovered or created, has been discussed for millennia. And the controversy is pretty much alive in the 21th century, as shown by a series of short papers published by the *Newsletter of the European Mathematical Society*, [14, 15, 16, 17, 18, 19, 20]. Lots of philosophical "theories" have been conceived to try to answer those puzzling questions. Although all of them do make pertinent and interesting remarks on these matters, none of them seems to quite yield completely satisfactory answers. Either they do not adequately explain how can mathematics have so many and quite impressive applications to the "real" world, or they do not clearly unravel in exactly what form are mathematical objects real.

But before we tackle this question of the status of mathematical objects, one should first try to make clear what exactly does one mean by "object". We have to be very careful here, since we humans have a more than natural tendency to attribute reality, or existence, overwhelmingly to material objects that our senses can directly detect. Now, even if one tries to restrict the notion of "real objects" to things that are "physical" in some sense, one immediately runs into some difficulties, as for example: are electromagnetic waves "real" objects? What about gravity? These do seem to have a form of existence that is quite different from, say, a rock.

But even a "physical" object like a person, for example[8],has layers of complexity that, although well known, are seldom thought of. To see this, let us consider some of the levels at which one can describe a human being (see Fig. 3) To a doctor, he is but a set of organs and its interrelations - let me draw here the reader's attention to the importance of these interrelations: rearranging the organs has absolutely dramatic consequences! Now, to a biologist, he is a set of cells and their (vital!) interrelations. To a physicist, he is but a set of atoms and their (crucial!) interrelations. But, and I find this extremely curious, according to the 1932 Nobel laureate in physics, Werner Heisenberg (1901--1976), elementary particles are "mathematical forms" ([6], p. 36) and, in general, (p. 51):

> The 'thing-in-itself' is for the atomic physicist, if he uses this concept at all, finally a mathematical structure.

Human being

↓

Set of organs and their interrelations

↓

Set of cells and their interrelations

↓

Set of atoms and their interrelations

↓

Mathematical entity?

Fig. 3: What is an object, really?

Therefore, the question of knowing what a "real" object is, in order to eventually help clarify what a mathematical object might be, leads us right into mathematical structures!

To complicate things even further, let us observe that a human being is a

[8] I do not believe in things for which there is no evidence for their actual existence, so I am here disregarding beliefs in the existence of supernatural (whatever that means!) components of humans or other animals.

set of atoms, together with their very special interrelationships, that varies with time! In an address to the National Academy of Sciences of the USA, in 1955, titled *The Value of Science* (included in [21], pp. 240--248), Richard Feynman[9] (1918--1988) noted:

> [the] phosphorus that is in the brain of a rat --- and also in mine, and yours --- is not the same phosphorus as it was two weeks ago. [...] the atoms that are in the brain are being replaced: the ones that were there before have gone away.
> So what is this mind of ours: what are these atoms with consciousness? Last week's potatoes! They now can remember what was going on in my mind a year ago --- a mind which has long ago been replaced. [...] the thing which I call my individuality is only a pattern or dance [...]. The atoms come into my brain, dance a dance, then go out --- there are always new atoms, but always doing the same dance, remembering what the dance was yesterday.

That is, humans and animals in general are much more like rivers than like rocks: they are patterns, rather than "fixed" physical objects.

From all of this, what I want here to emphasize is that relations between "physical" objects are as real and important as the objects themselves. And that there are laws and patterns ruling these interrelations which are as real as anything else. Mathematics seems to capture some of these inner relations that are just not visible to the naked eye. As Rudy Rucker, in p. 4 of [22], writes:

> Mathematics is the study of pure pattern, and everything in the cosmos is a kind of pattern.

Numbers themselves are but representations of some special kinds of relationships. To make this clear, let us first point out the distinction between a number, e.g. 6, and its representation[10]. In fact, "6" is not the number six (see Fig. 4).

Fig. 6: Several representations of the number 6

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[9] Nobel laureate in physics, 1965.

[10] The distinction between a representation and the thing being represented is eloquently illustrated by some paintings of René Magritte (1898--1967), namely the one titled "La trahison des images" (see http://en.wikipedia.org/wiki/The_Treachery_of_Images), which consists of a drawing of a pipe together with the sentence "this is not a pipe". This is entirely true: there is no pipe in the painting, only a representation of it!

So, what is the object represented by "6"? What does it refer to? More precisely, what exactly is the number six? Well, it is a certain "quantity", which is a certain property of a collection of objects. It is actually the common property of all collections that have that particular number of elements, and it captures a certain relation that those collections all have among themselves.

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx \qquad\qquad \sum_{n\geq 0}\frac{(-1)^n}{2n+1}$$

These are not mathematical objects

Fig. 5: Paraphrasing Magritte

In conclusion, mathematical objects encode some subtle relationships, and are not to be expected to literally exist out there in the same manner as a physical object exists, although one should be careful, since a detailed analysis shows that even "physical" objects may be quite more intricate than realized at first glance. So, numbers, triangles, circles, and other mathematical entities are but constructs that represent deep, hidden relations, and are not to be taken verbatim. But these relations are as real as any other "objects".

3. An Evolutionary Perspective

More than two millenia ago, Plato could not explain how humans seem to be born with some form of knowledge --- some sort of software, as we could now call it ---, and could not explain how can one reach previously unknown truths from deductions alone, except by arguing for the existence of another sort of parallel world, a world of "forms", and for the pre-existence of a "soul" that would have inhabited that world before being "attached" to a body. Now, most people do not seem to realize that those mysteries were solved, in a much more satisfactory way, about 150 years ago.

Before explaining how, let me rephrase what I tried to convey in the last section, that the "forms" of Plato do exist in this world, not in a mysterious and intangible ideal world. They are the laws governing the interconnections of matter and energy, and of more subtle properties, like "quantity" and various kinds of relations among things, and also the laws governing the interconnections between those "first-order" laws, maybe even some "higher-order" laws. They are all part of a sort of inner structure of the Cosmos. Mathematical objects (not their representations!) are the elements of that structure. But then how do we have access to that "mathematical structure"?

The answer was given by one of the most brilliant and diligent humans of all times: Charles Darwin (1809--1882), who perfectly summarized it in the so called "notebook M",[11] in which one can find (p. 128, in an entry dated 4 September 1838):

> Plato says in Phaedo that our "necessary ideas" arise from the preexistence of the soul, are not derivable from experience --- read monkeys for preexistence.

This is just a note that Darwin wrote to himself, but after one understands the history of life on this planet, which was made possible by the seminal discovery of "natural selection and descent with slow modification", its meaning becomes clear. We humans are the result of thousands of millions of years of selection, of real experiences made by countless generations of all our ancestors, from all the species of which we are descendants. There is therefore a vast array of experience contained in our genetic code, experiences that we draw upon to explore the Universe that surrounds us. So, when a human being is born is not some sort of blank slate, but comes equipped with powerful tools to understand Nature.

Now, the discovery of the mechanism of "natural selection and descent with slow modification"[12] or the "theory of evolution", as it is commonly known,[13] explained so many things that were previously completely baffling, and made intelligible an huge array of data and observations about living organisms previously scattered and mystifying. It stimulated, and continues to stimulate, fruitful research in several areas of biology.[14] However, after more than 150 years it is still not properly understood by many people, and there are too many misconceptions[15] and completely wrong ideas about it.

Among the main erroneous ideas that interfere with an understanding of the theory of evolution, let us mention the following: (a) life evolves purely randomly, which arises from not realizing that there is a sharp distinction between the randomness of mutations and the mechanism of natural

[11] Available online, at http://darwin-online.org.uk.
[12] [23]. See also [24, Chaps. 3 and 4.
[13] The term "evolution" is not the best, since it gives the wrong idea of a "progress", but unfortunately "natural selection and descent with slow modification" was just too big.
[14] See [25], Chaps. 7-10.
[15] See [26] and http://evolution.berkeley.edu/evolibrary/misconceptions_faq.php.

selection, which is anything but random; (b) to be "fit", meaning well adapted to a particular environment, implies to be ruthless and strong; (c) evolution implies a continuous progress from "inferior" animals to "superior" ones; (d) it justifies mean, cruel and immoral behaviour; (e) it justifies the "law of the jungle". Partly, these confusions came from the fact that there has been an exaggerated emphasis on competition over cooperation in descriptions and popular introductions of the theory of evolution. Here we limit ourselves to note that a human being is in fact the result of tremendously complicated symbiotic relationships. In our digestive tract alone there are hundreds of species of bacteria, essential to our survival, and their total number is ten times greater than the total number of human cells in the body[16]!

The wrong, but very pervasive, ideas about the theory of evolution, together with the fact that this theory removes humans from a central pedestal above all other living creatures (which hurts our natural anthropocentric feelings), lead to an emotional denial, be it conscious or unconscious, of the "transcendently democratic"[17]and profound consequences of the insights of Darwin. A perfect example of this, and quite relevant for the subject of this essay, is the following passage from [27][18] (p.19), where Roger Penrose clearly states why he prefers Plato's intangible, ideal world:

> How do I really feel about the possibility that all my actions, and those of my friends, are ultimately governed by mathematical principles of this kind? I can live with that. I would, indeed, prefer to have these actions controlled by something residing in some such aspect of Plato's fabulous mathematical world than to have them be subject to the kind of simplistic base motives, such as pleasure-seeking, personal greed, or aggressive violence, that many would argue to be the implications of a strictly scientific standpoint.

This shows that the author felt into the trap of some of the above mentioned misconceptions. It comes then as no surprise that he writes a little later:[19]

> it remains a deep puzzle why mathematical laws should apply to the world with such phenomenal precision.

In a Darwinian perspective this mystery starts to fade away, since, as Carl Sagan explains in [28], pp. 232--233:

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[16] See http://en.wikipedia.org/wiki/Gut_flora.
[17] See [24], p. 67.
[18] Which is, nevertheless, an amazing book, a true tour de force!
[19] [27], pp. 20--21.

> we can imagine a universe in which the laws of nature are immensely more complex. But we do not live in such a universe. Why not? I think it may be because all those organisms who perceived their universe as very complex are dead. Those of our arboreal ancestors who had difficulty computing their trajectories as they brachiated from tree to tree did not leave many offspring[20].Natural selection has served as a kind of intellectual sieve, producing brains and intelligences increasingly competent to deal with the laws of nature. This resonance, extracted by natural selection, between our brains and the universe may help explain a quandary set by Einstein: The most incomprehensible property of the universe, he said, is that it is so comprehensible.

I have always found it rather curious that everyone is so amazed with the extraordinary fine-tuning between some characteristics of some animals and their environment, and do not notice that the same applies to the human animal. They seem to assume, explicitly or, most of the time, implicitly, that there is a fundamental separation between our mental capabilities and Nature. The mind is the product of a natural selection that operated over a vast period of time, and is just as part of Nature as anything else. It contains remarkable adaptations of humans to their environment, including the capabilities of pattern detection, abstraction, and the organization of information. As Rudy Rucker so well puts it, in [22], p. 16:

> That our mathematics is effective for manipulating concepts is perhaps no more surprising than that our legs are good for walking.

4. The Source of Intuition

As the success of Physics in describing a huge amount of diverse phenomena shows, the Universe clearly seems to have a sort of inner mathematical "texture". Now, we humans are the product of a natural selection process that produced our brains, which can, through pattern recognition and abstraction, access at least part of that texture. This has allowed our species to uncover some parts of our Universe that totally escape detection by our senses, like radio waves, for instance. As sketched in Fig. 6, through an amazingly rich evolutive heritage, our brains are able to capture the mathematical structure of the Cosmos, and this has allowed us to enlarge our horizons, by uncovering parts of the Universe that were previously unknown to us.

[20] Obviously, this is just a caricatural example.

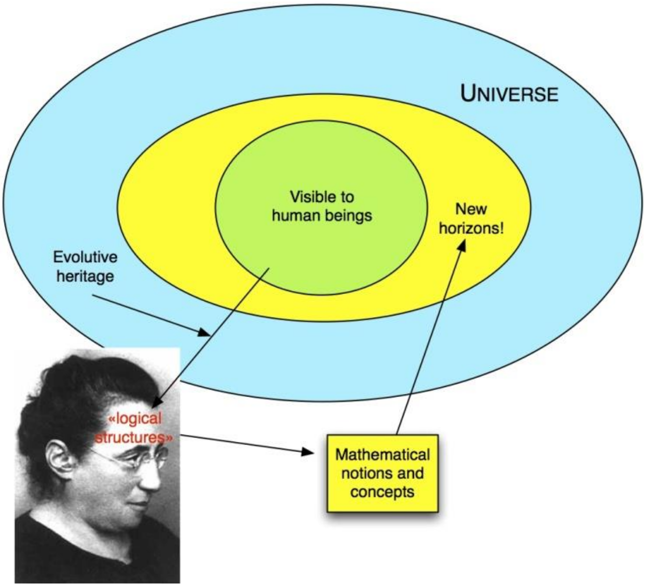

Fig. 6: The universe, human beings and mathematics

One can now see that working on an open problem in Mathematics, for example, helps to sharpen some of our main evolutionary tools, testing our intellectual limits, and this effort may lead to build the tools to overcome some of those limits. The evolutionary advantages of this should be obvious. Of course, any progress in any given problem will be but a tiny and very humble piece of knowledge about the intimate structure of our universe, but it is still worthwhile. Leonhard Euler (1707-1783) said it best [29]:

> knowledge of every truth is a worthy matter in itself, even of those which seem unrelated to popular use; we have seen that all truths, at least those which we are able to understand, are so greatly connected with one another, that we cannot consider any one of them altogether useless without some rashness.
>
> And so, even if a certain proposition seems to be this way, so that regardless of whether it turns out to be true or false, it would be of no benefit to us anyway, still the method itself, by which we would established its truth or falsity, nevertheless may be useful in opening up the way for us to discover other, more useful truths.

What I have tried to argue above is that, in order to understand what Mathematics is about, one must first realize that, besides physical objects (whatever they really are), and as importantly, the world contains some sort of intrinsic logical inner structure. And our brains have been selected to apprehend it, to some extent. Working on a mathematical problem, as abstract as it may be, is to uncover a tiny piece of that inner structure. The source of the intuitions that guide us in these investigations resides on our immensely rich evolutionary heritage.

Of course, how exactly does that genetic heritage comes alive in each one of us is still largely unknown, and to unravel its secrets will represent a tremendous challenge for generations to come. In the same vain, the problem of knowing precisely what is the mechanism behind mathematical intuitions, whatever exactly that means, the problem of knowing precisely what intuitions are behind a result like Proposition *P* discussed above, remains to be understood. But, and this has been the central point of this paper, one simply cannot do that without a proper evolutionary perspective.

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